Cubic interaction vertices in higher spin theories

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Abstract

Based purely on symmetry considerations, we derive the following result: in momentum space, the coefficient of the cubic interaction vertex for a spin λ field is equal to the corresponding Yang-Mills (spin 1) coefficient, raised to the power λ. This result is valid for all λ for Lagrangians that contain a cubic interaction vertex of the form λ-λ-λ, in four-dimensional flat spacetime. For $\lambda = 3$, we present an additional derivation of this result.

1 Introduction

A consistent description of the quantum interactions of higher spin fields ($\lambda > 2$) is frought with difficulties, mostly stemming from the higher-derivative structures inherent to such theories. The most comprehensive formulations of higher spin theories are purely on-shell [1], as opposed to off-shell descriptions [2] that are not fully understood even at the quartic interaction level. The infinite-dimensional gauge symmetries underlying massless higher spin theories could result in surprising ultra-violet properties, making them interesting to study. For a nice summary of various no-go theorems, associated with massless fields in flat spacetime, and ways around them, see [3]. Earlier off-shell investigations of interacting higher spin theories in flat space include [4, 5] and references therein.

In this paper, we derive the following result: In momentum space, the coefficient of the cubic interaction vertex for a spin λ field is equal to the corresponding Yang-Mills (spin=1) coefficient, raised to the power λ. This result is obtained only for Lagrangians that contain a cubic interaction vertex of the form λ-λ-λ, in four-dimensional flat spacetime. We briefly review the standard light-cone approach to constructing an interacting higher spin theory and then recast the results, introducing spinor helicity products, into a form that makes the result stated above, manifest. Light-cone gauge eliminates the unphysical degrees of freedom but Poincaré invariance is no longer manifest, and will be checked.

2 Cubic interaction vertices

We define light-cone co-ordinates in $(-,+,+,+)$ Minkowski space-time as

$$x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}} \ , \qquad x = \frac{x^1 + ix^2}{\sqrt{2}} \ , \qquad \bar{x} = \frac{x^1 - ix^2}{\sqrt{2}} \ . \tag{1}$$

The corresponding derivatives being $\partial_{\pm}$, $\bar{\partial}$ and ∂. In four spacetime dimensions, all massless fields have two physical degrees of freedom ϕ and $\bar{\phi}$. The generators of the Poincaré algebra, in light-cone coordinates, are the momenta [1]

$$p^- = i\frac{\partial\bar{\partial}}{\partial_-} = -p_+ \qquad p^+ = -i\partial^+ = -p_- \qquad p = -i\partial \qquad \bar{p} = -i\bar{\partial} \tag{2}$$

and the rotation generators

$$\begin{aligned} j &= (x\bar{\partial} - \bar{x}\partial - \lambda) \ , \qquad j^+ = i(x^+\partial - x\partial_-) \ , \\ j^{+-} &= i(x^+\frac{\partial\bar{\partial}}{\partial_-} - x^-\partial_-) \ , \qquad j^- = i(-x\frac{\partial\bar{\partial}}{\partial_-} - x^-\partial + \lambda\frac{\partial}{\partial_-}) \ , \end{aligned} \tag{3}$$

and their complex conjugates. λ is the helicity of the field and $\partial_+ = \frac{\partial\bar{\partial}}{\partial_-}$ using the free equations of motion (which picks up corrections in the interacting theory).

Any four-vector may be expressed as a bispinor using the Pauli matrices, $p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}}$, with $det(p_{a\dot{a}})$ yielding $-p^\mu p_\mu$. We also introduce the spinor product

$$< kl >= \sqrt{2}\frac{(kl_- - lk_-)}{\sqrt{k_- l_-}} \ . \tag{4}$$

[1] $\frac{1}{\partial_-}$ is defined using the prescription in [6].

The Hamiltonian for the free field theory is

$$H \equiv \int d^3x\, \mathcal{H} = -\int d^3x\, \bar{\phi}\, \partial\bar{\partial}\, \phi \ , \tag{5}$$

with the last equality valid only for the free theory. We also write

$$H \equiv \int d^3x\, \mathcal{H} = \int d^3x\, \partial_- \bar{\phi}\, \delta_{\mathcal{H}}\phi. \ , \tag{6}$$

thereby introducing the time translation operator

$$\delta_{\mathcal{H}}\phi \equiv \partial_+\phi = \{\phi, \mathcal{H}\} \tag{7}$$

where $\{,\}$ denotes the Poisson bracket. When interactions are switched on, this $\delta_{\mathcal{H}}$ operator picks up corrections, order by order in the coupling constant α. At order α, based purely on helicity considerations [2] , the interacting part of $\delta_{\mathcal{H}}$ has to have the form $\alpha\phi\phi + \alpha\bar{\phi}\phi$. For the first of these two terms, we start with the following ansatz

$$\delta^{\alpha}_{\mathcal{H}}\phi = \alpha\, \partial^{+\mu} \left[\bar{\partial}^a \partial^{+\rho}\phi \bar{\partial}^b \partial^{+\sigma}\phi \right] \ , \tag{8}$$

where μ, ρ, σ, a, b are integers. Three Poincaré generators pick up $O(\alpha)$ corrections

$$\begin{aligned}
\delta_{j^{+-}}\phi &= \delta^{0}_{j^{+-}}\phi - ix^+\delta^{\alpha}_{\mathcal{H}}\phi + O(\alpha^2) \ , \\
\delta_{j^{-}}\phi &= \delta^{0}_{j^{-}}\phi + ix\delta^{\alpha}_{\mathcal{H}}\phi + \delta^{\alpha}_{s}\phi + O(\alpha^2) \ , \\
\delta_{\bar{j}^{-}}\phi &= \delta^{0}_{\bar{j}^{-}}\phi + i\bar{x}\delta^{\alpha}_{\mathcal{H}}\phi + \delta^{\alpha}_{\bar{s}}\phi + O(\alpha^2) \ ,
\end{aligned} \tag{9}$$

where δ^{α}_{s} and $\delta^{\alpha}_{\bar{s}}$ represent spin transformations whose forms are not relevant to the results that follow. The requirement of closure of the Poincaré algebra imposes various conditions on the integers introduced in (8). The commutators

$$\begin{aligned}
[\delta_j, \delta^{\alpha}_{\mathcal{H}}]\,\phi &= 0 \ , \\
\left[\delta_{j^{+-}}, \delta^{\alpha}_{\mathcal{H}}\right]\phi &= 0 \ ,
\end{aligned} \tag{10}$$

yield

$$\begin{aligned}
a + b &= \lambda \ , \\
\mu + \rho + \sigma &= -1 \ ,
\end{aligned} \tag{11}$$

while the other commutation relations determine the values of a, b, μ, ρ and σ. We thus obtain [4]

$$\delta^{\alpha}_{\mathcal{H}}\phi = \alpha \sum_{n=0}^{\lambda} (-1)^n \binom{\lambda}{n} \partial^{+(\lambda-1)} \left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\phi \frac{\bar{\partial}^n}{\partial^{+n}}\phi \right] \ , \tag{12}$$

for even λ. For odd-helicity fields, closure of the algebra requires the introduction of an antisymmetric structure constant yielding

$$\delta^{\alpha}_{\mathcal{H}}\phi^a = \alpha f^{abc} \sum_{n=0}^{\lambda} (-1)^n \binom{\lambda}{n} \partial^{+(\lambda-1)} \left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\phi^b \frac{\bar{\partial}^n}{\partial^{+n}}\phi^c \right] \ . \tag{13}$$

[2]The field ϕ has helcity λ while the field $\bar{\phi}$ has helicity $-\lambda$.

We repeat the steps above for the $\alpha\bar{\phi}\phi$ term in $\delta_{\mathcal{H}}$. Using (6), we now obtain the complete Hamiltonian, to this order, and the corresponding Action [4]

$$S = \int d^4x \left(\frac{1}{2}\bar{\phi}\Box\phi + \alpha \sum_{n=0}^{\lambda}(-1)^n \binom{\lambda}{n} \bar{\phi}\partial^{+\lambda} \left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\phi \frac{\bar{\partial}^n}{\partial^{+n}}\phi \right] \right), \tag{14}$$

for even λ and

$$S = \int d^4x \left(\frac{1}{2}\bar{\phi}^a\Box\phi^a + \alpha f^{abc} \sum_{n=0}^{\lambda}(-1)^n \binom{\lambda}{n} \bar{\phi}^a\partial^{+\lambda} \left[\frac{\bar{\partial}^{(\lambda-n)}}{\partial^{+(\lambda-n)}}\phi^b \frac{\bar{\partial}^n}{\partial^{+n}}\phi^c \right] \right), \tag{15}$$

for odd λ.
Both cubic vertices above, in momentum-space, have the following structure (measure and constants suppressed)

$$\begin{aligned} &(\bar{k}_- + \bar{l}_-)^\lambda \delta^4(p+k+l) \sum_{n=0}^{\lambda}(-1)^n \binom{\lambda}{n} \left(\frac{\bar{k}}{k_-}\right)^{\lambda-n} \left(\frac{\bar{l}}{l_-}\right)^n \tilde{\bar{\phi}}(p)\tilde{\phi}(k)\tilde{\phi}(l) + c.c. \\ &= (\bar{k}_- + \bar{l}_-)^\lambda \delta^4(p+k+l) \left(\frac{\bar{k}l_- - \bar{l}k_-}{k_- l_-}\right)^\lambda \tilde{\bar{\phi}}(p)\tilde{\phi}(k)\tilde{\phi}(l) + c.c. \ . \end{aligned} \tag{16}$$

The momentum-conserving delta function $\delta^4(p+k+l)$ implies that

$$\begin{aligned} < \bar{l}\bar{p} >&= \sqrt{\frac{2}{\bar{p}_-\bar{l}_-}}(\bar{k}\bar{l}_- - \bar{l}\bar{k}_-) = \frac{\sqrt{\bar{k}_-}}{\sqrt{-(\bar{k}_- + \bar{l}_-)}} < \bar{k}\bar{l} > , \\ < \bar{p}\bar{k} >&= \sqrt{\frac{2}{\bar{p}_-\bar{k}_-}}(\bar{k}\bar{l}_- - \bar{l}\bar{k}_-) = \frac{\sqrt{\bar{l}_-}}{\sqrt{-(\bar{k}_- + \bar{l}_-)}} < \bar{k}\bar{l} > , \end{aligned} \tag{17}$$

allowing us to rewrite (14) and (15) as

$$S = \int \frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\ (2\pi)^4\delta^4(p+k+l)\ \left(\left[\frac{< \bar{k}\bar{l} >^3}{< \bar{l}\bar{p} >< \bar{p}\bar{k} >} \right]^\lambda \tilde{\bar{\phi}}(p)\tilde{\phi}(k)\tilde{\phi}(l) + c.c. \right) \tag{18}$$

for even λ and

$$S = \int \frac{d^4p}{(2\pi)^4}\frac{d^4k}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\ (2\pi)^4\delta^4(p+k+l)\ f^{abc} \left(\left[\frac{< \bar{k}\bar{l} >^3}{< \bar{l}\bar{p} >< \bar{p}\bar{k} >} \right]^\lambda \tilde{\bar{\phi}}^a(p)\tilde{\phi}^b(k)\tilde{\phi}^c(l) + c.c. \right) \tag{19}$$

for odd λ. This proves that the coefficient of the cubic interaction vertex in higher spin Lagrangians is equal to the corresponding coefficient in pure Yang-Mills theory raised to the power λ, ie.

$$L_3^\lambda = \left[\frac{\langle \bar{k}\,\bar{l}\rangle^3}{\langle \bar{l}\,\bar{p}\rangle \langle \bar{p}\,\bar{k}\rangle} \right]^\lambda . \tag{20}$$

3 Light-cone Action from the Covariant Action

In this section, we relate the light-cone and covariant approaches to higher spin theories for the case $\lambda = 3$. We derive (15), for $\lambda = 3$, starting from the covariant Action [10]. Our starting point is the free action

$$S_0 = \int d^4x \left(\frac{1}{2}\phi^a_{\mu\nu\rho}\Box\phi^{a\mu\nu\rho} + \frac{3}{2}\partial^\mu\phi^a_{\mu\sigma\rho}\partial_\nu\phi^{a\nu\sigma\rho} + \frac{3}{2}\partial_\mu\phi^a_\nu\partial^\mu\phi^{a\nu} + \frac{3}{4}\partial_\mu\phi^{a\mu}\partial_\nu\phi^{a\nu} - 3\partial_\mu\phi^a_\nu\partial_\rho\phi^{a\rho\mu\nu} \right) , \tag{21}$$

where $\phi^{a\mu} = \phi^{a\mu\nu}{}_\nu$. The corresponding field equations read

$$F^a_{\mu\nu\rho} - \frac{3}{2}\eta_{(\mu\nu}F^a_{\rho)} = 0 , \tag{22}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and

$$F^a_{\mu\nu\rho} = \Box\phi^a_{\mu\nu\rho} - 3\partial^\sigma\partial_{(\mu}\phi^a_{\nu\rho)\sigma} + 3\partial_{(\mu}\partial_\nu\phi^a_{\rho)} . \tag{23}$$

The Lagrangian and field equations are invariant under a gauge transformation of the form

$$\delta_\lambda\phi^a_{\mu\nu\rho} = 3\partial_{(\mu}\lambda^a_{\nu\rho)} . \tag{24}$$

$\lambda_{\nu\rho}$ is symmetric and traceless implying that nine gauge parameters may be chosen. Light-cone gauge is imposed by setting [11]

$$\phi^{+\mu\nu a} - \frac{1}{4}\eta^{\mu\nu}\phi^{+\sigma a}{}_\sigma = 0 . \tag{25}$$

The equations of motion yield

$$\begin{gathered}
\phi^{++-a} = \phi^{+11a} = \phi^{+22a} = 0 , \\
\phi^{-ija} = \frac{1}{\partial^+}\partial_k\phi^{kija} ; \qquad \phi^{--ia} = \frac{1}{\partial^{+2}}\partial_j\partial_k\phi^{ijka} , \\
\phi^{---a} = \frac{1}{\partial^{+3}}\partial_i\partial_j\partial_k\phi^{ijka} ; \qquad \phi^{11ia} = -\phi^{22ia} .
\end{gathered} \tag{26}$$

As a consequence, $\phi^{a\mu} = 0$ leaving only two independent components of $\phi^{\mu\nu\rho a}$ written as complex quantities $\phi^a = (2)^{-1/2}(\phi^{111a} + i\phi^{112a})$ and $\bar\phi^a = (2)^{-1/2}(\phi^{111a} - i\phi^{112a})$. Many terms in the interacting covariant action [10] involve $\phi^{\mu a}$ and hence vanish. The non-vanishing terms read

$$\begin{aligned}
S_\alpha = \frac{-3}{8}\alpha f^{abc}\int d^4x(&-2\partial_\delta\phi^a_{\rho\beta\gamma}\partial^\epsilon\phi^{b\rho\delta}{}_\sigma\phi^{c\sigma\beta\gamma}_\epsilon + \phi^a_{\rho\beta\gamma}\partial^\rho\phi^b_{\delta\epsilon}\partial^\beta\partial^\gamma\phi^{c\delta\epsilon\sigma} - 3\partial^\rho\phi^a_{\rho\beta\gamma}\partial^\epsilon\phi^{b\beta\gamma}_\delta\partial^\sigma\phi^c_{\epsilon\sigma}{}^\delta \\
&+3\partial^\rho\partial^\delta\phi^a_{\rho\beta\gamma}\partial^\sigma\phi^{b\beta\gamma\epsilon}\phi^c_{\epsilon\delta\sigma} + 6\partial_\delta\phi^a_{\rho\beta\gamma}\partial^\sigma\phi^{b\rho\beta\epsilon}\partial^\gamma\phi^c_{\epsilon\sigma}{}^\delta)
\end{aligned} \tag{27}$$

On expansion, the first, third and fourth terms vanish leaving us with

$$S_0 = \int d^4x\, 2\bar\phi^a\Box\phi^a + 8\left[2\left(\bar\phi^a\partial^{+3}\phi^b\frac{\bar\partial^3}{\partial^{+3}}\phi^c + 3\bar\phi^a\frac{\bar\partial^2}{\partial^{+2}}\phi^b\bar\partial\partial^{+2}\phi^c + 3\bar\phi^a\bar\partial^2\partial^+\phi^b\frac{\bar\partial}{\partial^+}\phi^c + \bar\phi^a\phi^b\bar\partial^3\phi^c\right)\right. \tag{28}$$

$$\left. +6\left(3\bar\phi^a\frac{\bar\partial^2}{\partial^+}\phi^b\partial^+\bar\partial\phi^c + 3\bar\phi^a\bar\partial^2\phi^b\bar\partial\phi^c + \bar\phi^a\partial^{+2}\phi^b\frac{\bar\partial^3}{\partial^{+2}}\phi^c + \bar\phi^a\partial^+\phi^b\frac{\bar\partial^3}{\partial^+}\phi^c\right) + c.c.\right]$$

which is (15) for $\lambda = 3$.

* * *

Although the cubic vertices in these theories have such neat structures it seems unlikely that they can be used in a straightforward manner to construct [8] higher order interaction vertices [9]. Further, could we expect relations of the form (20) to hold at higher orders in α? If so, do they define a consistent interacting tree-level S-matrix with the usual properties?

Acknowledgments

We thank Stefano Kovacs, Sunil Mukhi and Hidehiko Shimada for discussions. This work is supported by the Max Planck Institute for Gravitational Physics through the Max Planck Partner Group in Quantum Field Theory; the Department of Science and Technology, Government of India, through the Ramanujan (SA) and INSPIRE (YSA) fellowships.

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